

## NOTES AND CORRESPONDENCE

**Katabatic Flow: Analytic Solution for Gradually Varying Eddy Diffusivities**

BRANKO GRISOGONO AND JOHANNES OERLEMANS\*

*Department of Meteorology, Stockholm University, Stockholm, Sweden*

28 June 1999 and 8 December 2000

## ABSTRACT

A simple form of the Prandtl model addressing pure katabatic flows is solved. The new analytic solution is valid for almost any assigned eddy diffusivity  $K(z)$  and constant Prandtl number. This model assumes a one-dimensional steady state for momentum and heat balance. Its approximate solution, obtained using the WKB method, appears as a generalization and improvement of the classic analytic solution for the constant- $K$  case. It is compared favorably against a numerical solution. A comparison with observations from PASTEX, Austria 1994, shows that the new solution is much closer to the data than the constant- $K$  solution. The dynamics revealed with this new solution is discussed (relatively sharper near-surface profiles, their gradients, and the low-level jet), and a suggestion toward improving boundary layer parameterizations is offered.

**1. Introduction**

Recent interest in glacier fluctuations, sea level rise, and climate change has stimulated research on the atmospheric boundary layer over glacier and ice sheets. For instance, estimating changes in the mass balance of glaciers in response to climate changes requires a knowledge of the structure of this layer. Recent field experiments (e.g., Oerlemans and Vugts 1993; Greuell et al. 1997) have made clear that over land ice katabatic flow is very persistent, even on small valley glaciers. Evidently, the presence of katabatic flows greatly affects the surface fluxes of momentum, heat, and moisture (thus affecting glaciers as well). Analysis of data shows that standard profile analysis is usually not suitable to determine fluxes in conditions of katabatic flow (Munro 1989; Oerlemans 1998). More specifically, the analytic model of boundary layer slope winds (e.g., Prandtl 1942; Egger 1990) cannot provide sharp near-surface gradients that are observed (e.g., Munro 1989; Oerlemans 1998). Without these gradients, the calculated surface fluxes are wrong. A method has to be sought in which the coupling between dynamics and thermodynamics (buoyancy as a driving force of the flow) is

retained. Nappo and Rao (1987) show numerically that a pure katabatic flow is described by the Prandtl model, with vertically varying eddy diffusivity, in the limit of strong stability with vanishing advection and entrainment. Their and similar or more advanced higher-order closure numerical models (e.g., Arritt and Pielke 1986; Denby 1999) require such a fine resolution to model the low-level jet associated with katabatic flows, that they still remain in the research mode. Egger (1990) reviews flow models along cooled slopes; extensions and improvements of the Prandtl model either go toward two-dimensionality or numerical modeling. Weather prediction and climate models usually cannot afford the required resolution for pure katabatic flows thus calling for a parameterization, but current theories are insufficient for this task (Mahrt 1998). Here we tackle the mentioned problem by proposing an improved analytical solution to the pure katabatic flow.

In this paper we make a contribution by designing a 1D analytic model for the vertical structure of katabatic flow. As a starting point we use the mentioned classic model for slope winds of Prandtl (1942), applied to the katabatic flow over glaciers by Defant (1949). The problem is illustrated in Fig. 1 with an example of the katabatic flow accompanied with three constant eddy diffusion ( $K = \text{constant}$ ) solutions. The attempt is to fit the overall pattern observed, not a particular profile of the tower data. First, the value of  $K$  is changed. Second, the value of the Prandtl number (see later) is changed. It is obvious that the constant- $K$  model cannot reproduce sharp near-surface wind gradients seen in the data.

\* Permanent affiliation: Institute for Marine and Atmospheric Research, Utrecht University, Utrecht, Netherlands.

*Corresponding author address:* Dr. Branko Grisogono, Arrhenius-laboratory, Department of Meteorology, Stockholm University, S-10691 Stockholm, Sweden.  
E-mail: branko@misu.su.se

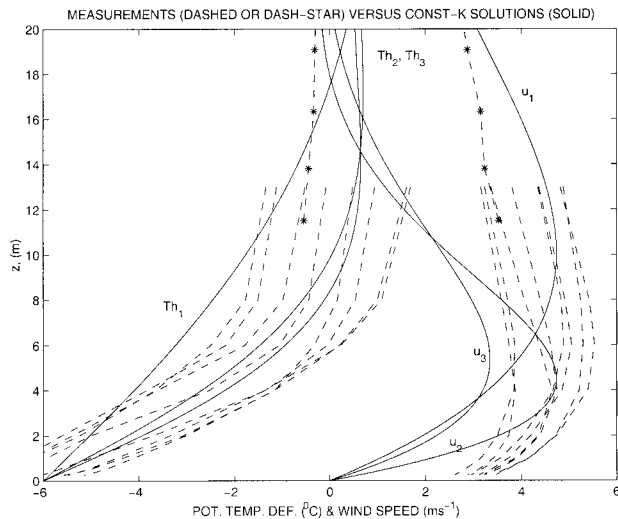


FIG. 1. A katabatic flow example. Potential temperature deficit  $\Theta$  and the related wind  $u$ : 24-h observation set from PASTEX (Austria, 27 Jul 1994, 30-min mean values for every 3 h, dashed), and the constant- $K$  solutions (solid). A balloon sounding is shown with dash-star curves. Diffusivity  $K_H = K = 0.06 \text{ m}^2 \text{ s}^{-1}$  for the Prandtl profile 1, and  $K = 0.012 \text{ m}^2 \text{ s}^{-1}$  for the Prandtl profiles 2 and 3. Prandtl number,  $\text{Pr} = 2$  for the Prandtl profiles 1 and 2, and  $\text{Pr} = 4$  for the profile 3. The lapse rate  $\gamma = 3 \text{ K km}^{-1}$ , slope  $\alpha = -0.1$ , surface temperature deficit  $C = -6^\circ\text{C}$  (see section 2a),  $\Theta_0 = 273.2 \text{ K}$ . [The units are in  $^\circ\text{C}$  or  $\text{m s}^{-1}$  (left and right, respectively) and height in m.]

Moreover, if the temperature profile is correct, the wind is wrong and vice versa. We shall return to this later.

The stable boundary layer (SBL) may contain the katabatic flow. In fact, over cold down-sloping surfaces it is the katabatic flow that often governs the SBL evolution (e.g., Oerlemans and Vugts 1993). Katabatic winds are frequently coupled with other terrain-induced flows (Vergeiner and Dreiseitl 1987); however, we will focus on a 1D model that is simple enough to be treated analytically but still relatively rich in its dynamics. A global solution will be sought for the linearized SBL problem based on the Prandtl model (e.g., Egger 1990); to find the potential temperature and wind profiles in a steady state for the given terrain slope  $\alpha$ , the background lapse rate  $\gamma$  and eddy diffusivity  $K(z)$ :

$$[\alpha, \gamma, K(z)] \rightarrow [\Theta(z), u(z)]. \quad (1.1)$$

Since the classic constant- $K$  approach provides inadequate near-surface profiles (e.g., Munro 1989; Oerlemans 1998), it appears natural to seek a method that allows for some sort of meaningful  $K$  variability. First, that would be a gradual  $K$  variability and that is attempted here. A singular perturbation method, a WKB technique, will be used (e.g., Bretherton 1969; Bender and Orszag 1978; Gill 1982). It contains the linear mathematical boundary layer theory as a special case. In practice, for an assigned gradually varying background field, perturbed linear fields,  $\Theta$  and  $u$ , will be computed. This “wave-type” method is seldom employed in geophysical boundary layer studies, probably because it has

to be assumed that background fields vary only gradually. The WKB approach offers simple, elegant, and revealing global solutions; these are usually in excellent agreement with the related numerical solutions—provided the WKB assumption is fulfilled (Laprise 1993; Grisogono 1995).

Apparently the problem (1.1) has not been solved yet in a general way. Here the well-known constant- $K$  solution will be extended to any gradually varying  $K(z)$ . The new result alters the structure of the constant- $K$  solution [e.g., near-surface  $\Theta(z)$  and  $u(z)$  will no longer change linearly] even though these are the same in the constant- $K$  limit. As the Ekman layer is unrealistic due to a constant  $K$  (yielding a controversy at its top and above with the required geostrophic balance) so is the classic Prandtl model. Besides a theoretical insight, it is hoped the solution provided here will be useful for data analyses of katabatic flows and for improvements of certain current SBL parameterizations when coupled with other parts of the boundary layer physics. According to Mahrt (1998), the problem tackled here belongs to unsolved types of the SBL because it contains a low-level jet.

## 2. The analytic model and its solution

### a. The governing equation and the constant- $K$ solution

For the pure katabatic flow (no background flow) there is, to a first order, a balance between the buoyancy force and frictional dissipation (e.g., van den Broeke 1997); this implies the model of Prandtl (1942). The governing 1D system pertaining to the Prandtl model and its assumptions are also described in Nappo and Rao (1987) and Egger (1990). A steady, Boussinesq, hydrostatic, irrotational flow with no pressure gradient is assumed and the  $K$  theory is invoked. The momentum and thermodynamic equations for the perturbations ( $\Theta$ ,  $u$ ) are

$$-g(\Theta/\Theta_0) \sin(\alpha) = d(K_M du/dz)/dz \quad (2.1a)$$

$$u\gamma \sin(\alpha) = d(K_H d\Theta/dz)/dz \quad (2.1b)$$

with the boundary conditions:  $\Theta(0) = C < 0$ ,  $\Theta(z \rightarrow \infty) = 0$ ,  $u(0) = 0$ ,  $u(z \rightarrow \infty) = 0$ , where the symbols have their usual meaning and the  $z$  axis is not vertical but perpendicular to the surface ( $x$  axis) sloped with angle  $\alpha$  from the horizontal. Note that  $\Theta$  is potential temperature deficit (i.e., the actual minus the background potential temperature, the latter also defining the constant potential temperature lapse rate  $\gamma$ ) and  $u$  is down the slope. Following Mahrt (1982), this is an equilibrium flow type (buoyancy acceleration retarded by turbulence stress divergence).

If  $K_M$  and  $K_H$  are flow independent, the Prandtl number  $\text{Pr}$  is constant,  $\text{Pr} \equiv K_M/K_H = \text{constant}$ , a linear fourth-order ordinary differential equation (ODE) results. For either dependent variable (here written for  $\Theta$  only) with  $K \equiv K_H$ , one obtains:

$$\Theta^{(4)} + 4K^{(1)}/K\Theta^{(3)} + f_2\Theta^{(2)} + f_1\Theta^{(1)} + (\sigma_0/K)^2\Theta = 0, \tag{2.2a}$$

where

$$\sigma_0^2 \equiv g\gamma \sin^2(\alpha)/(\Theta_0\text{Pr}) = (\text{buoyancy frequency})^2 \sin^2(\alpha)/\text{Pr}, \tag{2.2b}$$

$$f_1 \equiv K^{(3)}/K + K^{(1)}K^{(2)}/K^2, \tag{2.2c}$$

$$f_2 \equiv 3K^{(2)}/K + 2(K^{(1)}/K)^2,$$

with the superscript parentheses meaning the derivatives with respect to  $z$ , that is,  $( )^{(n)} = d^n( )/dz^n$ ,  $n = 1, 2, \dots$

The constant- $K$  solutions are revisited briefly; only the first and the last term in (2.2a) remain. The solution reads:

$$\Theta_{K=\text{const}} = C \exp\{-\sigma_c z\} \cos(\sigma_c z) \tag{2.3a}$$

$$u_{K=\text{const}} = -C\mu \exp\{-\sigma_c z\} \sin(\sigma_c z), \tag{2.3b}$$

where

$$\sigma_c^2 \equiv [g\gamma \sin^2(\alpha)/(4\Theta_0\text{Pr}K_H^2)]^{1/2} = \sigma_0/(2K_H) \tag{2.3c}$$

$$\mu \equiv [g/(\Theta_0\text{Pr}\gamma)]^{1/2}. \tag{2.3d}$$

A vertical-scale height is defined as the level where the wind ceases,

$$H_{u=0} = \pi/\sigma_c, \tag{2.3e}$$

in a more strict sense the katabatic flow is defined as that confined within the depth of  $1/\sigma_c$  (e.g., Oerlemans 1998). A compact form of (2.3a,b) is written as a dimensionless katabatic flow function:

$$F_C \equiv (\Theta_{K=\text{constant}} u_{K=\text{constant}})_{\text{D-LESS}} = \exp\{-(1+i)\sigma_c z\}. \tag{2.3f}$$

To convert into dimensional variables multiply  $\Theta$  or  $u$  (the real or imaginary part of  $F_C$  and later on  $F$ ) with  $C$  or  $-C\mu$ , respectively. The main problem with the solution (2.3) is that the near-surface profiles are linear with height; thus, sufficient gradients and adequate mixing cannot occur within the classic Prandtl model.

*b. The WKB method and the outer solution*

The WKB approach is justified if the background (assigned) quantities change more gradually than the calculated quantities. For example, if the vertical-scale height of the katabatic wind is about 10 m,  $K(z)$  may significantly vary only over more than that. We consider  $K(z) \geq 0$  having arbitrarily small values at the surface and higher up above the SBL, while reaching the maximum somewhere in the SBL. Now we divide the SBL into the outer region where  $K(z)$  generally decreases upward from its maximum,  $K_h$ , and the inner region where  $K(z)$  increases from  $K(z=0) = 0$  to the mentioned maximum  $K_h$  at the level  $h$ . Here the choice of  $K(z)$  is in agreement with O'Brien (1970); his  $K(z)$  has

been frequently used in modeling studies, see also Pielke (1984) or Stull (1988, Pp. 209–210). The third and fourth term in (2.2a) having  $f_1$  and  $f_2$  as factors are neglected henceforth (they would be kept in a higher-order WKB solution). Finally, the simplified governing ODE for  $F \equiv (\Theta, u)_{\text{D-LESS}}$  becomes

$$F^{(4)} + 4K^{(1)}/KF^{(3)} + [\sigma_0/K]^2 F = 0. \tag{2.4}$$

The outer WKB solution up to the first order for (2.4) is mathematically equivalent to that in Grisogono (1995) or in Berger and Grisogono (1998) for the Ekman layer. The first-order WKB solution reads

$$F \equiv F_{\text{OUT}} \sim [K(z)/K_h]^{-1/4} \exp\left\{- (1 \pm i)(\sigma_0/2)^{1/2} \int_0^z K^{-1/2} dz\right\}. \tag{2.5}$$

Although the integration in (2.5) goes formally from  $z = 0$ , this solution is valid only from the height  $h$ ,  $h \equiv h(K_h) = h[\max(K)]$ . Let us comment what (2.5) represents.

- Qualitative resemblance and generalization of the constant- $K$  solution, (2.3),
- A generalization of the vertical length-scale (2.3e):

$$\pi = (\sigma_0/2)^{1/2} \int_0^H K^{-1/2} dz. \tag{2.6}$$

It turns out that (2.3e) usually provides a somewhat different length scale  $H$  than (2.6) depending on which  $K = \text{constant}$  is chosen in the former and how  $K(z)$  decays with height in the latter. Aside from the parameters in  $\sigma_0$ , the form of  $K(z)$  determines  $H$  (but not opposite). An equivalent implicit formulation of  $H$  for the Ekman layer depth is found in Grisogono (1995). The obtained outer solution (2.5) describes the upper part of the SBL where the katabatic flow and  $K(z)$  decay with height and the surface inversion matches the background stability.

*c. The inner solution*

The inner WKB solution must satisfy the lower boundary conditions (giving the forcing) and smoothly approach the outer solution (no forcing). Fortunately, this can be fulfilled in a very simple way that has not, up to our knowledge, been obtained before. The inner region is characterized by  $K(z) \propto z +$  (higher-order terms),  $0 \leq z < h$  [this can be relaxed to  $K(z) \propto z^p$ ,  $0 \leq p < 2$ ]. Here the physical surface layer (SL) height  $h_s$  is a fraction of  $h(K_h)$ , typically:  $h_s < h \sim H/3$ , in accordance with O'Brien (1970), Stull (1988), etc. Thus, the inner solution comprises the SL and corresponds to an increasing  $K(z)$ .

Since  $K(z)$  is very small near the surface, a simple dominant balance analysis for (2.4) implies that its second term may be neglected. Ultimately then, only the

zero-order WKB solution for the approximate governing ODE

$$F^{(4)} + [\sigma_0/K]^2 F = 0 \tag{2.7}$$

will be considered, that is, the same governing ODE as in the constant- $K$  case. Although (2.7) comes out from a systematic perturbative analysis for the WKB, it could be obtained heuristically just by inspection. The zero-order WKB solution to (2.7) reads

$$F \equiv F_{\text{INN}} \sim \exp\left\{- (1 \pm i)(\sigma_0/2)^{1/2} \int_0^z K^{-1/2} dz\right\}, \tag{2.8}$$

and its main properties are the following:

- resemblance to the outer solution (but without the amplitude correction),
- stark contrast with the constant- $K$  solution near the surface;  $F_{\text{INN}}$  shows rapid growth for  $0 < z \ll h$ .

Clearly, if

$$K \propto z + \dots \Rightarrow \int_0^z K^{-1/2} dz \propto z^{1/2}$$

in (2.8), then

---


$$F \sim \begin{cases} \exp\left\{- (1 \pm i)(\sigma_0/2)^{1/2} \int_0^z K^{-1/2} dz\right\}, & z \leq h \\ [K(z)/K_h]^{-1/4} \exp\left\{- (1 \pm i)(\sigma_0/2)^{1/2} \int_0^z K^{-1/2} dz\right\}, & z \geq h, \end{cases} \tag{2.10}$$


---

where we will choose  $(1 \pm i) \rightarrow (1 + i)$ , to have the previous relation between  $F$  and  $(\Theta, u)$ . If  $K = \text{constant}$ , then (2.10) yields the constant- $K$  solution (2.3). If  $K \neq \text{constant}$ , cumulative effects of  $K(z)$  change the amplitude (e.g., the inversion strength and the low-level wind maximum) and the phase (e.g., the inversion and wind maximum positions) of the solution. Above the height  $h$ , the amplitude is slightly corrected by the  $[K(z)/K_h]^{-1/4}$  factor that contributes usually up to several percent of  $F$ . While the imaginary part of the constant- $K$  and WKB solutions reach their maxima at  $\pi/4$ , the corresponding real parts reach maxima at  $3\pi/4$ . Note that  $\pi/4$  will usually correspond to a height within the SL where the zero-order WKB solution is used; thus, the constant- $K$  and the WKB wind maxima shall typically be equal, although probably reached at different heights. On the contrary, the other maxima, at  $3\pi/4$ , will typically conform to a height above  $h$  where the first-order WKB solution applies, that is, the one with the amplitude correction; hence, the constant- $K$  and the WKB temperature maxima generally do not coincide. The WKB and constant- $K$  solutions both suggest somewhat different scale heights for  $\Theta$  and  $u$ .

$$u \sim \text{Im}\{F_{\text{INN}}\} \sim \exp\{-(\sigma z/2)^{1/2}\} \sin[(\sigma z/2)^{1/2}] \sim (\sigma z/2)^{1/2} + \dots \tag{2.9a}$$

$$\Theta \sim \text{Re}\{F_{\text{INN}}\} \sim \exp\{-(\sigma z/2)^{1/2}\} \cos[(\sigma z/2)^{1/2}] \sim 1 - (\sigma z/2)^{1/2} + \dots \tag{2.9b}$$

For small  $z$ , (2.9) grows rapidly compared to the equivalent expansion of the constant- $K$  solution (2.3f) that grows only linearly with small  $z$  (note  $\sigma \propto \sigma_0$ ). Consequently,  $F_{\text{INN}}$  exhibits sharp gradients that qualitatively relate to the psi functions correcting the neutral near-surface profiles toward stratification effects but all for horizontal surfaces. Hence, (2.9) is qualitatively consistent with the SL theories and observations of the stable horizontal surface layer (e.g., Stull 1988).

#### d. Solution for the whole SBL

Here  $F_{\text{OUT}}$  (the first-order WKB) and  $F_{\text{INN}}$  (the zeroth-order WKB) are combined in a global solution for the whole SBL. The patching, which is a simplified, localized matching (e.g., Bender and Orszag 1978), of (2.5) and (2.8) at  $h = h(K_h)$  gives the solution that is the main result of this study:

The necessity of  $K(z)$  assignment and its scale requirements are the weaknesses of this approach. Here is no feedback from the katabatic flow to  $K(z)$ . However, some preliminary numerical tests indicate a possibility for a realistic and efficient time-dependent coupling between the flow and  $K$ . This calls for a numerical approach.

#### e. A parameterization remark

Once  $F$  is obtained analytically, its derivative can be calculated directly. This  $dF/dz$  relates to the momentum and heat flux (e.g., Stull 1988). Using the roughness concept and certain further assumptions, these fluxes could be extended down to the surface providing the surface flux parameters  $u_*$  and  $\theta_*$ . Following van der Avoird and Duynkerke (1999), we extrapolate the momentum flux downward linearly from the low-level jet while the heat flux is kept constant from the jet height (Grisogono and Oerlemans 2001). These parameters,  $u_*$  and  $\theta_*$ , are essential for any boundary layer and they could be used, for example, for estimating changes in the mass balance of glaciers (important for climate monitoring).

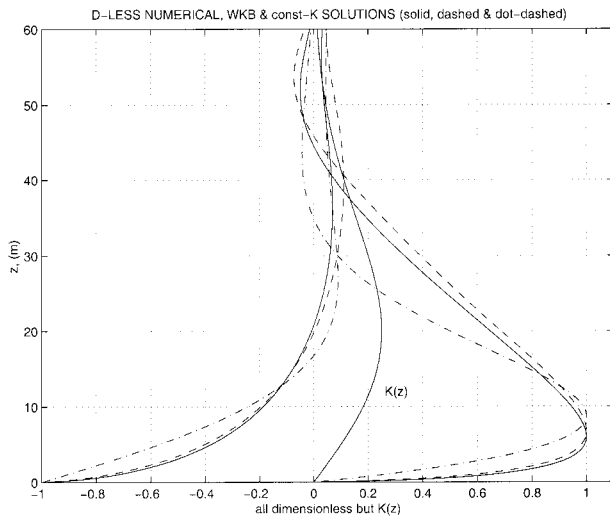


FIG. 2. Dimensionless idealized katabatic flow profiles of  $\Theta(z)$  and  $u(z)$ , left and right, from the proposed solution (2.10) vs the constant- $K$  and numerical solutions (2.3), dashed, dot-dashed, and solid, respectively. Here  $K = K_H$  (also solid) is of an O'Brien type, with the maximum of  $0.25 \text{ m}^2 \text{ s}^{-1}$  at  $h = 20 \text{ m}$ , and is not normalized. The constant- $K$  solution is based on  $0.3 \max[K(z)] = 0.075 \text{ m}^2 \text{ s}^{-1}$ ,  $\text{Pr} = 1.1$  (a larger  $\text{Pr}$  deepens the wind profile). The  $\Theta$  and  $u$  are normalized by  $|C|$  and  $\max(u)$ , respectively.

### 3. Examples

First, an idealized example is briefly discussed to gain more insight into the new solution (2.10). Second, a comparison with the real katabatic flow, introduced in Fig. 1, is given. A comparison among our WKB solution, a numerical one, and the constant- $K$  solution is shown in Fig. 2. The best choice for the constant- $K$  value seems to be around  $0.3 \max[K(z)]$ , less than the mean  $K$  averaged over  $H$ , provided  $K(0) = K(\infty) = 0$ . The numerical solution<sup>1</sup> should be the most accurate one. The mentioned properties of the WKB solution are revealed: the stronger and sharper near-surface inversion and the wind maximum occurring at a lower level. It also agrees well with the numerical solution. If  $K = \text{constant}$  is decreased further (to approximate the near-surface gradients), the related SBL becomes too thin. The third-order polynomial  $K(z) = P_3(z)$  by O'Brien (1970) is often used to model eddy diffusivity (Pielke 1984; Stull 1988). It is generalized here into a linear-exponential function that, if expanded for small  $z$ , approximates the O'Brien  $P_3(z)$ . Hence, the assumed  $K(z)$  profile is

$$K(z) = \text{constant } z \exp\{-0.5(z/h)^2\}. \quad (3.1)$$

This  $K(z)$  is used for its analytical tractability, but a couple of others are tried too, like  $K \sim z(1 - z/H)^2$ , and similar results are obtained. The PASTEX dataset is used to test the new solution. The experiment took

<sup>1</sup> The original system (2.1) with (3.1) is solved as a time-dependent problem relaxed to its steady state. The third-order Adams–Bashforth scheme centered in space is used.

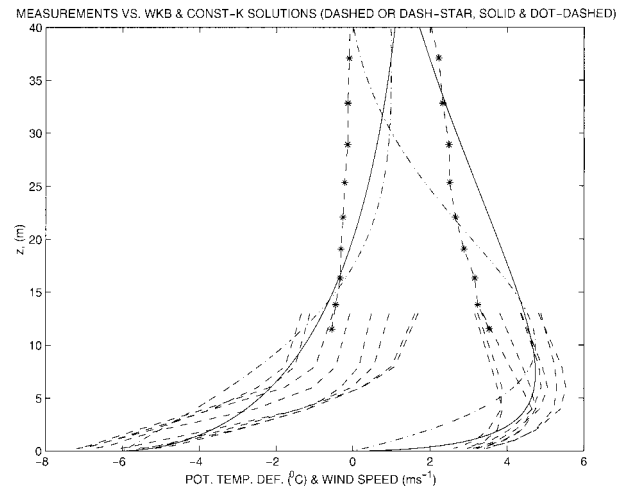


FIG. 3. Data comparison for  $\Theta$  (potential temperature deficit) and  $u$  (katabatic wind) among the WKB solution (solid) and the observations (dashed and dash-star), with the constant- $K$  solution (dot-dashed) from Fig. 1. The generalized O'Brien  $K(z)$  is used, Eq. (3.1), with  $\max[K(z)] = 0.30 \text{ m}^2 \text{ s}^{-1}$  at  $h = 30 \text{ m}$ ; other parameters:  $\gamma = 3 \text{ K km}^{-1}$ ,  $\alpha = -0.1$ ,  $C = -6^\circ\text{C}$ ,  $\text{Pr} = 2$  [as in Fig. 1 with profile 1 having  $K = 0.2 \max(K(z))$ ]. [The units are  $^\circ\text{C}$  or  $\text{m s}^{-1}$  (left and right, respectively) and height in m.]

place on the Pasterze glacier, Austria, in the summer of 1994 (e.g., van den Broeke 1997; Greuell et al. 1997; Oerlemans 1998). The glacier surface melting guarantees a simple lower boundary condition, that is, constant surface temperature.

Figure 3, similar to Fig. 1 but now with the WKB solution included, displays the comparison. The other data also suggests clear examples of katabatic flows with insignificant background or some valley flow (not shown, see, e.g., van den Broeke 1997). Based on the data and after several trials, the input is:  $\gamma = 3 \text{ K km}^{-1}$ ,  $\alpha = -0.1$ ,  $C = -6^\circ\text{C}$ ,  $\text{Pr} = 2$ ,  $\Theta_0 = 273.2 \text{ K}$ ,  $\max[K(z)] = 0.30 \text{ m}^2 \text{ s}^{-1}$  at  $h = 30 \text{ m}$ . Note in Fig. 3 the low-level mixing and the wind maximum around 5-m height, respectively, present in the data and the new WKB solution; also, the near-surface inversion strength and its location are captured reasonably well. The observed extreme values, their positions, and the overall shapes (e.g., the upper part of the low-level jet) are well simulated with the new but not with the constant- $K$  solution.

### 4. Concluding remarks

To understand the coupling between the atmosphere and cool, inclined surfaces, we must better understand the katabatic flow. An analytic step toward this coupling is attempted here. To summarize, the Prandtl model for katabatic flows is solved for gradually varying  $K(z)$  and constant but arbitrary Prandtl numbers. The new steady-state solution, obtained with the WKB method and also checked numerically, is a natural generalization of the known constant- $K$  solution, and it carries on a more realistic dynamics. A comparison with observations from PASTEX, Aus-

tria, 1994, shows that the new solution resembles the data much better than the constant- $K$  solution.

The proposed solution has its structure similar to the constant- $K$  solution. Essentially, the difference with the constant- $K$  solution can be summarized in two points: 1) the solution includes an integral, inherent to the WKB method, involving  $K(z)$  instead of a fixed  $K$  value, and 2) the solution consists of two parts, derived with the same technique (WKB) that are easily combined. Hence, the relative simplicity of the new solution results while gaining on the dynamics appreciably.

A drawback of our approach is a prescription of  $K(z)$  instead of having  $K$  as a function of the katabatic flow,  $K(\Theta, u)$ . That is why this study belongs to linear analytic modeling, that is, midway to nonlinear (and almost exclusively) numerical modeling. Another critical point is the way of prescribing  $K(z)$ . The WKB method makes a restriction on the allowed background variability, that is,  $K(z)$  may vary only on the scale that is larger than that for  $\Theta$  and  $u$ .

Following Mahrt (1998) and van der Avoird and Duynkerke (1999), to describe the SBL with a katabatic flow, it is essential to represent its low-level jet properly. The proposed solution does that well thus offering a new possibility for flux estimations. Suggestions for a new SBL parameterization might emerge after including the roughness concept and other parts of the surface physics, for example, the surface energy balance. By prescribing  $K(z = 0) \rightarrow K(z_0) \equiv K_* > 0$ , (a new lower boundary value for  $K$  at the roughness length  $z_0$ ), the WKB solution would be suited for such a coupling. The amplitude correction for the inner solution, if desired, might be of a type

$$(K_*/K)^{1/4} \{ 1 + [K^2/(K_*K_h)]^{1/4} - (K/K_h)^{1/4} \}.$$

Thus, a sound surface flux parameterization scheme would be coupled by merely giving an appropriate  $K_*$ . Preliminary results for the surface momentum flux,  $K_*(du/dz)|_0$  are encouraging. Those calculations are much needed for understanding of air-ice interactions (e.g., Kuhn 1979; Oerlemans 1998), improving numerical model parameterizations, etc. Also left for the future is a more systematic data comparison throughout the SBL.

*Acknowledgments.* Our colleagues at the Departments of Meteorology at Utrecht, Netherlands; Stockholm, Sweden, and Uppsala Universities are thanked for discussions. Constructive criticisms by Leif Enger, Erlend Källén, Sergej Zilitinkevich, and Michael Tjernström are appreciated. J.O. thanks the IMI for providing the resources for his stay in Sweden.

#### REFERENCES

- Arritt, R. W., and R. A. Pielke, 1986: Interactions of nocturnal slope flows with ambient winds. *Bound.-Layer Meteor.*, **37**, 183–195.

- Bender, C. M., and S. A. Orszag, 1978: *Advanced Mathematical Methods for Scientists and Engineers*. McGraw-Hill, 593 pp.
- Berger, B. W., and B. Grisogono, 1998: The baroclinic, variable eddy viscosity Ekman layer. An approximate analytical solution. *Bound.-Layer Meteor.*, **87**, 363–380.
- Bretherton, F. P., 1969: Momentum transport by gravity waves. *Quart. J. Roy. Meteor. Soc.*, **95**, 213–243.
- Defant, F., 1949: Zur theorie der Hangwinde, nebst bemerkungen zur Theorie der Berg- und Talwinde. *Arch. Meteor. Geophys. Biokl.*, **A1**, 421–450.
- Denby, B., 1999: Second-order modelling of turbulence in katabatic flows. *Bound.-Layer Meteor.*, **92**, 67–100.
- Egger, J., 1990: Thermally forced flows: Theory. *Atmospheric Processes over Complex Terrain*, W. Blumen, Ed., Amer. Meteor. Soc., 43–57.
- Gill, A. E., 1982: *Atmosphere–Ocean Dynamics*. Academic Press, 662 pp.
- Greuell, W., W. Knap, and P. Smeets, 1997: Elevational changes in meteorological variables along a midlatitude glacier during summer. *J. Geophys. Res.*, **102**, 25 941–25 954.
- Grisogono, B., 1995: A generalized Ekman layer profile within gradually-varying eddy diffusivities. *Quart. J. Roy. Meteor. Soc.*, **121**, 445–453.
- , and J. Oerlemans, 2001: A theory for the estimation of surface fluxes in simple katabatic flows. *Quart. J. Roy. Meteor. Soc.*, in press.
- Kuhn, M., 1979: On the computation of heat transfer coefficients from energy-balance gradients on a glacier. *J. Glaciol.*, **22**, 263–272.
- Laprise, J. P. R., 1993: An assessment of the WKB approximation to the vertical structure of linear mountain waves: Implications for gravity-wave drag parameterization. *J. Atmos. Sci.*, **50**, 1469–1487.
- Mahrt, L., 1982: Momentum balance of gravity flows. *J. Atmos. Sci.*, **39**, 2701–2711.
- , 1998: Stratified atmospheric boundary layers and breakdown of models. *Theor. Comput. Fluid Dyn.*, **11**, 263–279.
- Munro, D. S., 1989: Surface roughness and bulk heat transfer on a glacier. *J. Glaciol.*, **35**, 343–348.
- Nappo, C. J., and K. S. Rao, 1987: A model study of pure katabatic flow. *Tellus*, **39A**, 61–71.
- O'Brien, J. J., 1970: A note on the vertical structure of the eddy exchange coefficient in the planetary boundary layer. *J. Atmos. Sci.*, **27**, 1213–1215.
- Oerlemans, J., 1998: The atmospheric boundary layer over melting glaciers. *Clear and Cloudy Boundary Layers*, Royal Netherlands Academy of Arts and Sciences, 129–153.
- , and H. F. Vugts, 1993: A meteorological experiment in the melting zone of the Greenland ice sheet. *Bull. Amer. Meteor. Soc.*, **74**, 355–365.
- Pielke, R. A., 1984: *Mesoscale Numerical Modeling*. Academic Press, 612 pp.
- Prandtl, L., 1942: *Führer durch die Strömungslehre*. Vieweg und Sohn, 648 pp.
- Stull, R. B., 1988: *An Introduction to Boundary Layer Meteorology*. Kluwer Academic, 666 pp.
- van den Broeke, M. R., 1997: Momentum, heat, and moisture budgets of the katabatic wind layer over a midlatitude glacier in summer. *J. Appl. Meteor.*, **36**, 763–774.
- van der Avoird, E., and P. G. Duynkerke, 1999: Turbulence in a katabatic flow. Does it resemble turbulence in stable boundary layers over flat surfaces? *Bound.-Layer Meteor.*, **92**, 39–66.
- Vergeiner, I., and E. Dreiseitl, 1987: Valley winds and slope winds—Observations and elementary thoughts. *Meteor. Atmos. Phys.*, **36**, 264–286.